Nonlinear dynamics of the magnetization in an anisotropic ferromagnet with a magnetic field

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Introducing a particular parameter in the equations of motion for the magnetization in an anisotropic ferromagnet with a magnetic field, the Lax equations for Darboux matrices are generated recursively, the Jost solutions are satisfied the corresponding Lax equations, and the nonlinear dynamics of the magnetization are investigated. The results show that the solitary waves depend essentially on two velocities which describe a spin configuration deviating from a homogeneous magnetization. The center of inhomogeneity moves with a constant velocity, while the shape of solitary waves also changes with another velocity. The depths and widths of surface of solitary waves vary periodically with time, meanwhile its shapes are not symmetrical with respect to the center. The z component of the total magnetic moment and the total magnetic moment are not constants. The asymptotic behavior of multisoliton solutions is also analyzed. [S1063-651X(96)12710-0]

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I. INTRODUCTION

The study of ferromagnets is of considerable intrinsic interest, especially from the points of view of both soliton theory and condensed matter physics [1–6]. In particular, its continuum limit is governed by the Landau-Lifschitz equation, and it displays fascinating geometrical aspects: isotropic [7–10] and pure anisotropic [11–13] systems are geometrically equivalent, and gauge equivalent to a nonlinear Schrödinger equation. These, as well as the biaxial anisotropic [14–19] systems, are completely integrable. On the experimental side, a ferromagnet with an easy plane in a symmetry-breaking external transverse field has received continuing interest, though the most theoretical treatments have been based on the approximate mapping [20] to a sine-Gordon equation.

By separating variables in the moving coordinates, Tjio and Wright [21] and Quispel and Capel [22] separately obtained the Landau-Lifschitz equation for an isotropic ferromagnet and a ferromagnet with an easy axis. In terms of an inverse scattering transformation, Takhtajan [23] outlined very briefly the main steps of the solution of equation of motion. Fogedby [24] gave the detains of the procedure mentioned. Unfortunately, some essential steps in his arguments, such as the estimation of the value of Jost solutions, can hardly be accepted with satisfaction. Pu, Zhou, and Li [25] reported exclusively the multisoliton solutions of the Landau-Lifshitz equation. However, the Landau-Lifschitz equation for a ferromagnet with an easy plane was previously unsolved [2]. It is impossible to find the general stationary solution, as mentioned by Tjio and Wright [21].

Reducing the equation of motion to a sine-Gordon equation for a ferromagnet with an easy plane, Mikeska [26] obtained a solution. However, there exist some questions about this approach. First, this reduction has not been rigorously established except for $T\rightarrow 0$. Then, it is apart from the quantum effects [2], which are particularly crucial for $CsNiF_3$ with $S=1$. Third, it is inadequate [27], as shown by the neutron scattering experiments in $CsNiF_3$. Finally, when an external field tends to zero, this solution becomes a traveling wave which does not obviously relate to nonlinearity of spin interactions. Long and Bishop [28] proposed another solution. However, when an anisotropic approach vanished this solution does not tend to the well-known solution of an isotropic ferromagnet. Using the variation method, Nakumura and Sasada [11] obtained a solution. If this solution is directly substituted into the equation of motion, it does not satisfy this equation. Reducing the equation of motion to an appropriate form, Kosevich Ivanov, and Kovalev [29] found a solution. But it could not be considered as an approximate solution of equation for a ferromagnet with an easy plane, since it does not satisfy this equation even in the approximation of first order anisotropy.

Borisov [30] and Sklyanin [31] have formulated separately an inverse scattering problem in its classical form, i.e., in terms of equations of the Marchenko type for a complete

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anisotropic ferromagnet. By means of an inverse scattering transformation, Mikhailov [32] and Rodin [16] were able to reduce the problem to Riemann boundary value problem on a torus. However, these results are expressed by the elliptic function, they are more complicated, and they are difficult to transform to those of ferromagnet with an easy plane. Even though soliton solutions were found, they are difficult to transform to those of a ferromagnet with an easy plane, as mentioned by Faddeev and Takhtajan [3]. Derivating the Marchenko equation by an inverse scattering transformation, Borovik and co-workers [33,34] could not find even the single soliton solution in a uniaxial anisotropic ferromagnet. Using the Hirota method, Bogdan and Kovalev [35] attempted to construct exact multisoliton solutions in an anisotropic ferromagnet. However, they could not prove a series of nontrivial identities for the parameters of the solution. When an easy plane anisotropy is weak, explicit expressions cannot be obtained. Taking into account only the first order approximation, Ivanov, Kosevich, and Babich [36] obtained a useful result.

There exist some difficulties in the study of the nonlinear dynamics of the magnetization in an anisotropic ferromagnet. Its equation of motion, differing from those of an isotropic ferromagnet, could not be solved by the method of separating variables in moving coordinates [21,22]. Then this equation could not be solved by an inverse scattering transformation; in addition to complexity due to the Riemann surface, there is the double-valued function of the standard spectral parameter, and the reflection coefficient at the edges of cuts in the complex plane could not be neglected even in the case of nonreflection. Therefore, an exact treatment of the nonlinear dynamics of the magnetization in an anisotropic ferromagnet has never been done to our knowledge.

It is the purpose of this paper to investigate exactly the nonlinear dynamics of the magnetization in an anisotropic ferromagnet with a magnetic field. This paper is organized as follows: in Sec. II introducing a particular parameter, the Lax equations for Darboux matrices are generated recursively. Section III shows that Jost solutions satisfy the corresponding Lax equations. The exact soliton solutions are obtained, and it is shown that the z component of the total magnetic moment and the total magnetic momentum are not constants. In Sec. IV the asymptotic behavior of multisoliton solutions is also analyzed. Section V contains conclusions. This approach is a good method of studying of the nonlinear dynamics of the magnetization, in the case of a ferromagnet with anisotropy in the presence of an external magnetic field.

II. EQUATIONS OF MOTION

In the macroscopic theory of ferromagnetism, the magnetic state of a crystal is described by the magnetization vector $\mathbf{M}=(M_x,M_y,M_z)$, while the dynamics and kinetics of a ferromagnet are determined by variations of its magnetization. As a function of space coordinates and time, the magnetization of a ferromagnet $\mathbf{M}(x,t)$ is a solution of the Landau-Lifschitz equation

$$\frac{\partial \mathbf{M}}{\partial t}=\frac{2\mu_B}{\hbar}\mathbf{M}\times\frac{\delta E}{\delta \mathbf{M}}, \tag{1}$$

where μ_B is the Bohr magneton. Equation (1) has an integral of motion $<\mathbf{M}^2>\equiv\mathbf{M}_0^2=\text{const}$. In the ground state, the quantity M_0 coincides with a so-called spontaneous magnetization $M_0=(2\mu_B S/a^3)$, where S is the atomic spin and a is the interatomic spacing.

In general, the magnetic energy E of a biaxial anisotropic ferromagnet, including an exchange energy E_{ex}, a magnetic anisotropic energy E_{an}, and a Zeeman energy E_Z, can be written as

$$\begin{aligned} E&=E_{\text{ex}}+E_{\text{an}}+E_Z\\ &=\tfrac{1}{2}\alpha\int\sum_k\frac{\partial \mathbf{M}}{\partial x_k}\frac{\partial \mathbf{M}}{\partial x_k}d^3x-\tfrac{1}{2}\beta_x\int M_x^2d^3x\\ &\quad-\tfrac{1}{2}\beta_z\int M_z^2d^3x-\mu_B\int \mathbf{M}\cdot\mathbf{B}d^3x. \end{aligned} \tag{2}$$

If $E_{\text{an}}=0$, a crystal is called an isotropic ferromagnet. In the limit $\beta_x=0$, a biaxial anisotropic ferromagnet changes into an uniaxial anisotropic ferromagnet: when $\beta_z>0$, an anisotropy is of an easy-axis type, and when $\beta_z<0$ it is of an easy-plane type.

If measuring the space coordinate x and time coordinate t in units of $l_0=(J/\beta_z)^{1/2}$ and $\omega_0=(2\mu_B\beta_z M_0/\hbar)$, then according to Eqs. (1) and (2), we can obtain the following equation of motion:

$$\partial_t\mathbf{M}=\mathbf{M}\times[\partial_{xx}\mathbf{M}+J\mathbf{M}+\mu_B\mathbf{B}], \tag{3}$$

where the matrix $J=\text{diag}\,(J_x,J_y,J_z)$ is related to the anisotropic constants. In particular, we may choose $J=\text{diag}(0,\zeta,\rho^2+\zeta)$, where $\zeta=|\beta_x|\beta_z(\beta_x<0)$. In the case of a uniaxial anisotropic ferromagnet, $J=\text{diag}(0,0,\rho^2)$. In general, the third term on the right-hand side of Eq. (3) describe various perturbations such as an external field in this paper. When an external field is longitudinal, $\mathbf{B}=(0,0,B^z)$, this term can be removed by the gauge transformation, so that the system becomes integrable. However, if an external field is transverse, e.g., $\mathbf{B}=(B^x,0,0)$, this term is not removable by the gauge transformation, and none of the spin components remain conserved quantities. Consequently, the combined Galilean plus gauge invariance of the equation is broken, and no Lax pairs seem to exist; this system is generally thought to be nonintegrable. Only in the absence of either an anisotropic interaction or an external field does this system become integrable. When the oscillations of the magnetization vector $\mathbf{M}$ are localized near an easy plane yz, Eq. (3) has transformed a sine-Gordon equation in the limit $J_x<<J_y<J_z$. Similarly, this equation also changes a nonlinear Schrödinger equation in the limit $J_x\approx J_y<<J_z$, when the oscillations of the vector $\mathbf{M}$ are localized in the vicinity of the vacuum state $\mathbf{M}(x,t)=(0,0,M_0)$.

Equation (3) may be represented as a compatibility condition $\partial_t L-\partial_x A+[L,A]=0$ of two equations for 2×2 matrices $\Psi(x,t;\mu,\lambda)$:

$$\begin{aligned} \partial_x\Psi(x,t;\mu,\lambda)&=L(\mu,\lambda)\Psi(x,t;\mu,\lambda),\\ \partial_t\Psi(x,t;\mu,\lambda)&=A(\mu,\lambda)\Psi(x,t;\mu,\lambda), \end{aligned} \tag{4}$$

while

$$L(\mu,\lambda)=-i\mu(M_x\sigma_x+M_y\sigma_y)-i\lambda M_z\sigma_z,$$

$$A(\mu,\lambda)=-i\mu(M_y\partial_xM_z-M_z\partial_xM_y)\sigma_x-i\mu(M_z\partial_xM_x-M_x\partial_xM_z)\sigma_y-i\lambda(M_x\partial_xM_y-M_y\partial_xM_x)\sigma_z+i2\mu\lambda(M_x\sigma_x+M_y\sigma_y)+i2\mu^2M_z\sigma_z, \quad (5)$$

where $\sigma_\alpha(\alpha=x,y,z)$ are the Pauli matrices.

Since the parameters λ and μ in Eq. (5) satisfy the relation $\lambda^2=\mu^2-4\eta^2$, where $\eta^2=\rho^2+\mu_BB$. If one of them is taken to be an independent parameter, then the other is a double-valued function of the first, and it is then necessary to introduce a Riemann surface. In order to avoid the complexity brought about by a Riemann surface, we will introduce a particular parameter ξ,

$$\lambda=\xi-\eta^2\xi^{-1}, \quad \mu=\xi+\eta^2\xi^{-1}, \quad (6)$$

where $\xi=\pm\eta$ correspond to zero λ and to $\mu=\pm2\eta$. In the complex μ plane, these two points are the edges of cuts. This indicates that the edges of cuts must make a contribution even in the case of nonreflection when we use an inverse scattering transformation. The corresponding Lax equations are written as

$$\partial_x\Psi(\xi)=L(\xi)\Psi(\xi),$$
$$\partial_t\Psi(\xi)=A(\xi)\Psi(\xi). \quad (7)$$

There are two different types of the physical boundary conditions for Eq. (3). The boundary condition of the first type, corresponding to breatherlike solutions usually called magnetic solitons, is chosen as

$$\mathbf{M}\rightarrow\mathbf{M_0}=(0,0,M_0) \quad \text{at } x\rightarrow\pm\infty. \quad (8)$$

The corresponding Jost solution of Eq. (7) may be chosen as

$$\Psi_0(\xi)=\tfrac{1}{2}\{I-i(\sigma_x+\sigma_y+\sigma_z)\}\exp\left\{-i(\xi-\eta^2\xi^{-1})\times M_0\left[x-2\frac{(\xi^2+\eta^2)^2}{\xi(\xi^2-\eta^2)}t\right]\sigma_z\right\}. \quad (9)$$

One of the most powerful methods for constructing exact solutions of nonlinear integrable equations is the Darboux transformation method [37–42]. Using Darboux matrices $D_n(\xi)$, we can define the Jost solution $\Psi_n(\xi)$ of Eq. (7),

$$\Psi_n(\xi)=D_n(\xi)\Psi_{n-1}(\xi), \quad (10)$$

where $n=1,2,3,\ldots$, $D_n(\xi)$ has two poles ξ_n and $-\overline{\xi_n}$. Substituting Eq. (10) into Eq. (7) with a suitable subscript, the Lax equations for $D_n(\xi)$ can be written as

$$\partial_xD_n(\xi)=L_n(\xi)D_n(\xi)-D_n(\xi)L_{n-1}(\xi),$$
$$\partial_tD_n(\xi)=A_n(\xi)D_n(\xi)-D_n(\xi)A_{n-1}(\xi). \quad (11)$$

In Appendix A, we will obtain some relations for the Lax pairs L and A, the Jost solutions $\Psi_0(\xi)$ and $\Psi_n(\xi)$, and the Darboux matrices $D_n(\xi)$. They are useful for further calculations in the rest of this paper.

III. SOLITONS

When $D_n(\xi)$ has only two simple poles ξ_n and $-\overline{\xi_n}$, we can define

$$D_n(\xi)=C_nB_n(\xi), \quad (12)$$

where C_n is a 2×2 matrix independent of ξ, and

$$B_n(\xi)=I-\frac{\xi_n-\overline{\xi_n}}{\xi_n-\xi}F_n-\frac{\overline{\xi_n}-\xi_n}{\overline{\xi_n}+\xi}\widetilde{F}_n, \quad (13)$$

while

$$(\xi_n-\overline{\xi_n})C_nF_n, \quad (\xi_n-\overline{\xi_n})C_n\widetilde{F}_n \quad (14)$$

are residues at poles ξ_n and $-\overline{\xi_n}$, where F_n and $\widetilde{F}_n$ are also 2×2 matrices independent of ξ, respectively.

The following are relations for Darboux matrices $D_n(\xi)$:

$$D_n^\dagger(\overline{\xi})=B_n^\dagger(\overline{\xi})C_n^\dagger, \quad (15)$$

$$D_n^{-1}(\xi)=B_n^{-1}(\xi)C_n^{-1}, \quad (16)$$

and

$$D_n(\xi)D_n^{-1}(\xi)=D_n^{-1}(\xi)D_n(\xi)=I. \quad (17)$$

In the rest of this section, we will determine $B_n(\xi)$ and C_n separately. First, according to Appendix A and Eq. (15), one can obtain the following relations for $B_n(\xi)$:

$$B_n^\dagger(\overline{\xi})=I-\frac{\overline{\xi_n}-\xi_n}{\overline{\xi_n}-\xi}F_n^\dagger-\frac{\xi_n-\overline{\xi_n}}{\xi_n+\xi}\sigma_xF_n^T\sigma_x \quad (18)$$

and

$$B_n^{-1}(\xi)=B_n^\dagger(\overline{\xi}), \quad (19)$$

where the superscript T means transpose, while

$$\widetilde{F}_n=\sigma_x\overline{F_n}\sigma_x. \quad (20)$$

Since $D_n(\xi)D_n^{-1}(\xi)=D_n^{-1}(\xi)D_n(\xi)=I$ in Eq. (17), it has not poles, i.e., $F_nB_n^\dagger(\xi_n)=0$, i.e.,

$$F_n\left(I-F_n^\dagger-\frac{\xi_n-\overline{\xi_n}}{2\xi_n}\sigma_xF_n^T\sigma_x\right)=0; \quad (21)$$

this result shows that F_n is degenerated.

In Appendix B, we will obtain F_n and $\widetilde{F}_n$ separately. In terms of Eq. (13) and Appendix B, $B_n(\xi)$ can be expressed by

$$B_n(\xi)=[(\xi-\xi_n)(\xi_+\overline{\xi_n})(\overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2)(\overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2)]^{-1}\begin{pmatrix}\overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2 & 0\\ 0 & \overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2\end{pmatrix}$$
$$\times\left[\xi^2\begin{pmatrix}\overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2 & 0\\ 0 & \overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2\end{pmatrix}+\xi(\xi_n^2-\overline{\xi_n^2})\begin{pmatrix}0 & \overline{\gamma_n}\delta_n\\ \overline{\delta_n}\gamma_n & 0\end{pmatrix}\right.$$
$$\left.-|\xi_n|^2\begin{pmatrix}\overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2 & 0\\ 0 & \overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2\end{pmatrix}\right]. \tag{22}$$

Second, by means of Appendix A and Eq. (12), there are the following relations for C_n:

$$C_n=\sigma_x\overline{C_n}\sigma_x, \tag{23}$$

$$C_n^\dagger=C_n^{-1}, \tag{24}$$

and

$$C_nC_n^\dagger=I; \tag{25}$$

this result shows that C_n is a diagonal, i.e.,

$$(C_n)_{12}=(C_n)_{21}=0, \tag{26}$$

$$(C_n)_{11}=\overline{(C_n)_{22}}, \tag{27}$$

and

$$|(C_n)_{11}|=1. \tag{28}$$

Since only the module of $(C_n)_{11}$ is equal to 1, one can write

$$C_n=\exp\left(\frac{i}{2}\theta_n\sigma_z\right), \tag{29}$$

where θ_n is real and characteristic of the rotation angle of spin in the xy plane; it may be dependent on x and t.

In order to determine C_n, substituting Eq. (14) into Eq. (11), then taking the limit $\xi\to\infty$ and 0, we can obtain

$$\partial_x(C_n)=-i2\eta(\mathbf{M}_n)_z\sigma_z(C_n)+(C_n)i2\eta(\mathbf{M}_n)_z\sigma_z, \tag{30}$$
$$\partial_x(C_nB_n(0))=i2\eta(\mathbf{M}_n)_z\sigma_z(C_nB_n(0))-[C_nB_n(0)]i2\eta(\mathbf{M}_n)_z\sigma_z.$$

Comparing these two equations, one can find

$$C_n^{-2}=B_n(0). \tag{31}$$

Using Eqs. (22) and (31), C_n can be determined by

$$C_n=[(\overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2)(\overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2)]^{-1/2}\times\begin{pmatrix}\overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2 & 0\\ 0 & \overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2\end{pmatrix}, \tag{32}$$

while θ_n in Eq. (29) can be written as

$$\theta_n=2\tan^{-1}\left[\frac{\xi_n''(|\gamma_n|^2-|\delta_n|^2)}{\xi_n'(|\gamma_n|^2+|\delta_n|^2)}\right], \tag{33}$$

where ξ_n' and ξ_n'' denote the real and imaginary part of ξ_n, respectively.

Up to now, we have obtained C_n and $B_n(\xi)$, i.e., the Darboux matrices $D_n(\xi)$ have been recursively determined. Substituting Eqs. (22) and (32) into Eq. (12), $D_n(\xi)$ can be expressed by

$$D_n(\xi)=\{(\xi-\xi_n)(\xi_+\overline{\xi_n})[(\overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2)(\overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2)]^{3/2}\}^{-1}$$
$$\times\begin{pmatrix}(\overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2)(\overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2) & 0\\ 0 & (\overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2)(\overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2)\end{pmatrix}$$
$$\times\left[\xi^2\begin{pmatrix}\overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2 & 0\\ 0 & \overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2\end{pmatrix}+\xi(\xi_n^2-\overline{\xi_n^2})\begin{pmatrix}0 & \overline{\gamma_n}\delta_n\\ \overline{\delta_n}\gamma_n & 0\end{pmatrix}\right.$$
$$\left.-|\xi_n|^2\begin{pmatrix}\overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2 & 0\\ 0 & \overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2\end{pmatrix}\right]. \tag{34}$$

In order to determine γ_n and δ_n, substituting Eq. (14) into Eq. (11), then taking the limit $\xi\to\xi_n$, Eq.(11) can be written as

$$\partial_x(C_n F_n \Psi_{n-1}(\xi_n)) = L_n(\xi_n)[C_n F_n \Psi_{n-1}(\xi_n)],$$
$$\partial_t(C_n F_n \Psi_{n-1}(\xi_n)) = A_n(\xi_n)[C_n F_n \Psi_{n-1}(\xi_n)], \tag{35}$$

where the factor is independent of x and t. Because F_n is the degeneracy, the second factor of the right-hand side, i.e., $(\gamma_n \delta_n)\Psi_{n-1}(\xi_n)$, should appear in the left-hand side with its original form; therefore, we can simply obtain

$$(\gamma_n \delta_n) = (b_n 1)\Psi_{n-1}^{-1}(\xi_n). \tag{36}$$

When b_n is a constant, it will be determined by the boundary condition and the initial condition.

When $\xi \to 1$, according to Eq. (11) and Appendix A, we can obtain

$$(\mathbf{M}_n \cdot \sigma) = D_n(1)(\mathbf{M}_{n-1} \cdot \sigma) D_n^{\dagger}(1), \tag{37}$$

where $D_n(1)$ can be written as

$$D_n(1) = [(1-\xi_n)(1+\overline{\xi_n})(\overline{\xi_n}|\gamma_n|^2 + \xi_n|\delta_n|^2)(\overline{\xi_n}|\delta_n|^2 + \xi_n|\gamma_n|^2)]^{-1} \times \begin{pmatrix} (\overline{\xi_n}|\delta_n|^2 + \xi_n|\gamma_n|^2)(\overline{\xi_n}|\gamma_n|^2 + \xi_n|\delta_n|^2) & 0 \\ 0 & (\overline{\xi_n}|\gamma_n|^2 + \xi_n|\delta_n|^2)(\overline{\xi_n}|\delta_n|^2 + \xi_n|\gamma_n|^2) \end{pmatrix} \times \begin{pmatrix} (1-\xi_n^2)\overline{\xi_n}|\delta_n|^2 + (1-\overline{\xi_n^2})\xi_n|\gamma_n|^2 & (\xi_n^2 - \overline{\xi_n^2})\overline{\gamma_n}\delta_n \\ (\xi_n^2 - \overline{\xi_n^2})\overline{\delta_n}\gamma_n & (1-\xi_n^2)\overline{\xi_n}|\gamma_n|^2 + (1-\overline{\xi_n^2})\xi_n|\delta_n|^2 \end{pmatrix}. \tag{38}$$

Similarly, when $\xi \to -1$, in terms of Eq. (11) and Appendix A, we can also obtain

$$\sigma_z(\mathbf{M}_n \cdot \sigma)\sigma_z = D_n(-1)\sigma_z(\mathbf{M}_{n-1} \cdot \sigma)\sigma_z D_n^{\dagger}(-1). \tag{39}$$

Using Appendix A and $C_n C_n^{+} = I$ in Eq. (25), Eq.(39) can be transformed into

$$\sigma_z(\mathbf{M}_n \cdot \sigma)\sigma_z = -\sigma_x \overline{(\mathbf{M}_n \cdot \sigma)}\sigma_x . \tag{40}$$

When $n=1$, according to Eqs. (37) and (40), we can obtain

$$(\mathbf{M}_1)_x - i(\mathbf{M}_1)_y = (D_1(1))_{12}\overline{(D_1(1))_{21}} + (D_1(1))_{11}\overline{(D_1(1))_{22}}, \tag{41}$$

$$(\mathbf{M}_1)_z = (D_1(1))_{12}\overline{(D_1(1))_{11}} + (D_1(1))_{11}\overline{(D_1(1))_{12}}, \tag{42}$$

where $D_1(1)$ can be written as

$$D_1(1) = [(1-\xi_1)(1+\overline{\xi_1})(\overline{\xi_1}|\gamma_1|^2 + \xi_1|\delta_1|^2)(\overline{\xi_1}|\delta_1|^2 + \xi_1|\gamma_1|^2)]^{-1} \times \begin{pmatrix} (\overline{\xi_1}|\delta_1|^2 + \xi_1|\gamma_1|^2)(\overline{\xi_1}|\gamma_1|^2 + \xi_1|\delta_1|^2) & 0 \\ 0 & (\overline{\xi_1}|\gamma_1|^2 + \xi_1|\delta_1|^2)(\overline{\xi_1}|\delta_1|^2 + \xi_1|\gamma_1|^2) \end{pmatrix} \times \begin{pmatrix} (1-\xi_1^2)\overline{\xi_1}|\delta_1|^2 + (1-\overline{\xi_1^2})\xi_1|\gamma_1|^2 & (\xi_1^2 - \overline{\xi_1^2})\overline{\gamma_1}\delta_1 \\ (\xi_1^2 - \overline{\xi_1^2})\overline{\delta_1}\gamma_1 & (1-\xi_1^2)\overline{\xi_1}|\gamma_1|^2 + (1-\overline{\xi_1^2})\xi_1|\delta_1|^2 \end{pmatrix}. \tag{43}$$

In terms of Eq. (36), since only relative values of $(b_n 1)$ have meaning, one can find

$$(\gamma_1 \quad \delta_1) \sim (f_1 \quad f_1^{-1})\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \tag{44}$$

where

$$f_1 = b_1^{1/2}\exp\left\{ i(\xi_1 - \eta^2\xi_1^{-1})\left[x - 2\frac{(\xi_1^2 + \eta^2)^2}{\xi_1(\xi_1^2 - \eta^2)} t \right] \right\}; \tag{45}$$

therefore

$$\gamma_1 = f_1 + i f_1^{-1}, \quad \delta_1 = f_1 - i f_1^{-1}, \tag{46}$$

while

$$f_1 = \exp(-\phi_1 + i\phi_2), \tag{47}$$

$$\phi_1 = \frac{\xi_1''(|\xi_1|^2 + \eta^2)}{|\xi_1|^2}(x - V_1 t - x_{10}), \tag{48}$$

and

$$\phi_2=\frac{\xi_1'(|\xi_1|^2-\eta^2)}{|\xi_1|^2}(x-V_2t-x_{20}), \tag{49}$$

where

$$V_1=\frac{4\xi_1'(|\xi_1|^2-\eta^2)}{|\xi_1|^2} \tag{50}$$

and

$$V_2=\frac{2(\xi_1'^2-\xi_1''^2)(|\xi_1|^4+\eta^4)+4\eta^2|\xi_1|^4}{\xi_1'|\xi_1|^2(|\xi_1|^2-\eta^2)}. \tag{51}$$

By means of Eqs. (41)–(51), the single soliton solutions can be written as

$$(\mathbf{M}_1)_x=\frac{2\xi_1''^2(|\xi_1|^4-\eta^4)\sinh\phi_1\sin\phi_2+2\xi_1'\xi_1''(|\xi_1|^2-\eta^2)^2\cosh\phi_1\cos\phi_2}{|\xi_1|^2(|\xi_1|^2-\eta^2)^2\cosh^2\phi_1+4\eta^2\xi_1''^2|\xi_1|^2\sin^2\phi_2}, \tag{52}$$

$$(\mathbf{M}_1)_y=\frac{2\xi_1''^2(|\xi_1|^2-\eta^2)^2\sinh\phi_1\cos\phi_2-2\xi_1'\xi_1''(|\xi_1|^4-\eta^4)\cosh\phi_1\sin\phi_2}{|\xi_1|^2(|\xi_1|^2-\eta^2)^2\cosh^2\phi_1+4\eta^2\xi_1''^2|\xi_1|^2\sin^2\phi_2}, \tag{53}$$

$$(\mathbf{M}_1)_z=M_0-\frac{2\xi_1''^2(|\xi_1|^2-\eta^2)^2+8\eta^2\xi_1''^2|\xi_1|^2\sin^2\phi_2}{|\xi_1|^2(|\xi_1|^2-\eta^2)^2\cosh^2\phi_1+4\eta^2\xi_1''^2|\xi_1|^2\sin^2\phi_2}. \tag{54}$$

Similarly, we can also obtain the two-soliton, three-soliton, and multisoliton solutions.

It is concluded that the solitary waves (52)–(54) depend essentially on two velocities V_1 in Eq. (50) and V_2 in Eq. (51), which describe a spin configuration deviating from a homogeneous magnetization. The center of an inhomogeneity moves with a constant velocity V_1, while the shape of solitary waves (the direction of magnetization in its center) also changes with another velocity V_2.

In the polar coordinates, taking the z axis as the polar axis,

$$\cos\theta=1-\frac{2\xi_1''^2(|\xi_1|^2-\eta^2)^2+8\eta^2\xi_1''^2|\xi_1|^2\sin^2\phi_2}{|\xi_1|^2(|\xi_1|^2-\eta^2)^2\cosh^2\phi_1+4\eta^2\xi_1''^2|\xi_1|^2\sin^2\phi_2}, \tag{55}$$

$$\varphi=\varphi_0+\phi_2+\tan^{-1}\left[\frac{\xi_1''(|\xi_1|^2-\eta^2)}{\xi_1'(|\xi_1|^2+\eta^2)}\tanh\phi_1\right]+2\tan^{-1}\left[\frac{2|\xi_1|^2\eta^2}{|\xi_1|^4-\eta^4}\tanh\phi_1\right], \tag{56}$$

we can find the following property:

$$\cos\theta(-x,-t)=\cos\theta(x,t). \tag{57}$$

In order to analyze the features of the previous soliton solutions, setting the preliminary values as zero in the moving coordinates of the soliton,

$$\cos\theta=1-\frac{2\xi_1''^2(|\xi_1|^2-\eta^2)^2+8\eta^2\xi_1''^2|\xi_1|^2\sin^2\left[\frac{\xi_1'(|\xi_1|^2-\eta^2)}{|\xi_1|^2}(x-V_2t)\right]}{|\xi_1|^2(|\xi_1|^2-\eta^2)^2\cosh^2\left[\frac{\xi_1''(|\xi_1|^2+\eta^2)}{|\xi_1|^2}x\right]+4\eta^2\xi_1''^2|\xi_1|^2\sin^2\left[\frac{\xi_1'(|\xi_1|^2-\eta^2)}{|\xi_1|^2}(x-V_2t)\right]}, \tag{58}$$

$$\varphi=\varphi_0+\frac{\xi_1'(|\xi_1|^2-\eta^2)}{|\xi_1|^2}(x-V_2t)+\tan^{-1}\left\{\frac{\xi_1''(|\xi_1|^2-\eta^2)}{\xi_1'(|\xi_1|^2+\eta^2)}\tanh\left[\frac{\xi_1''(|\xi_1|^2+\eta^2)}{|\xi_1|^2}x\right]\right\}$$
$$+2\tan^{-1}\left\{\frac{2|\xi_1|^2\eta^2}{|\xi_1|^4-\eta^4}\tanh\left[\frac{\xi_1''(|\xi_1|^2+\eta^2)}{|\xi_1|^2}x\right]\right\}. \tag{59}$$

Therefore, the depths and widths of the surface of solitary waves are not constants, but vary periodically with time. The shape of the solitary waves also changes with velocity V_2, and it is not symmetrical with respect to the center. This feature did not appear in the soliton solution for all othernonlinear equations solved.

Obviously, when $\eta\to0$, $\mu=\lambda$, and these soliton solutions in an anisotropic ferromagnet reduce to those in an isotropic ferromagnet; for example, the single soliton solutions (52)–(54) are transformed to

$$(\mathbf{M}_1)_x = \frac{2\xi_1''}{|\xi_1|^2}\mathrm{sech}^2[\xi_1''(x-4\xi_1't-x_{10})]\Bigg(\xi_1''\sinh[\xi_1''(x-4\xi_1't-x_{10})]\sin\left\{\xi_1'\left[x-2\left(\xi_1'-\frac{\xi_1''^2}{\xi_1'}\right)t-x_{20}\right]\right\} + \xi_1'\cosh[\xi_1''(x-4\xi_1't-x_{10})]\cos\left\{\xi_1'\left[x-2\left(\xi_1'-\frac{\xi_1''^2}{\xi_1'}\right)t-x_{20}\right]\right\}\Bigg), \tag{60}$$

$$(\mathbf{M}_1)_y = \frac{2\xi_1''}{|\xi_1|^2}\mathrm{sech}^2[\xi_1''(x-4\xi_1't-x_{10})]\Bigg\{\xi_1''\sinh[\xi_1''(x-4\xi_1't-x_{10})]\cos\left[\xi_1'\left(x-2\left(\xi_1'-\frac{\xi_1''^2}{\xi_1'}\right)t-x_{20}\right)\right] - \xi_1'\cosh[\xi_1''(x-4\xi_1't-x_{10})]\sin\left[\xi_1'\left(x-2\left(\xi_1'-\frac{\xi_1''^2}{\xi_1'}\right)t-x_{20}\right)\right]\Bigg\}, \tag{61}$$

$$(\mathbf{M}_1)_z = M_0 - \frac{2\xi_1''^2}{|\xi_1|^2}\mathrm{sech}^2[\xi_1''(x-4\xi_1't-x_{10})]. \tag{62}$$

These results are equal to Eq. (27a) obtained by the method of an inverse scattering transformation in Ref. [25]. While taking the z axis as the polar axis in the polar coordinates,

$$\cos\theta = 1 - \frac{2\xi_1''^2}{|\xi_1|^2}\mathrm{sech}^2[\xi_1''(x-4\xi_1't-x_{10})], \tag{63}$$

$$\varphi = \varphi_0 + \xi_1'\left[x-2\left(\xi_1'-\frac{\xi_1''^2}{\xi_1'}\right)t-x_{20}\right] + \tan^{-1}\left\{\frac{\xi_1''}{\xi_1'}\tanh[\xi_1''(x-4\xi_1't-x_{10})]\right\}. \tag{64}$$

When $t\to 0$, these results are equivalent to Eq. (22) obtained by means of the method of the separating variables in the moving coordinates in Ref. [21].

Figures 1–3 give some graphical illustrations of a previous soliton solution $(\mathbf{M}_1)_z$ expressed by Eq. (54) in an anisotropic ferromagnet, and that by Eq. (62) in an isotropic ferromagnet. In these figures, we took the parameters $\xi_1'=0.1$, $\xi_1''=0.2$, $x_{10}=0$, $x_{20}=0$, and $\pi/(4V_1)$ as a unit of time in three figures, then set $\eta=0.10$ in Fig. 1, $\eta=0.33$ in Fig. 2, and $\eta=0$ in Fig. 3. If the $x-(\mathbf{M}_1)_z$ plane is taken as a reference plane when $t=0$, we can directly find the following feature of solitary wave $(\mathbf{M}_1)_z$.

(1) Since the lowest point of the surface is located in the plane of the center of surface, we can observe the motion of center by looking at the motion of the lowest point. The lowest point of the surface in the previous figures moves with three constant velocities V_1 corresponding to three anisotropic parameters η, respectively.

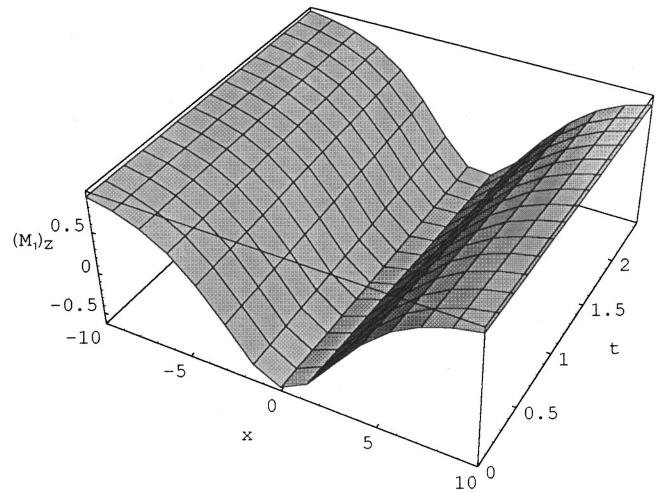

FIG. 1. Some graphical illustrations of a soliton solution $(\mathbf{M}_1)_z$ expressed by Eq. (54) in an anisotropic ferromagnet, where $\eta=0.10$, $\xi_1'=0.1$, $\xi_1''=0.2$, $x_{10}=0$, $x_{20}=0$, and $\pi/(4V_1)$ as units of time.

(2) The shape of the surface of $(\mathbf{M}_1)_z$ changes with another constant velocity V_2; the surface is not symmetrical with respect to the center. When $\eta\to 0$, the soliton solution $(\mathbf{M}_1)_z$, expressed by Eq. (54) in an anisotropic ferromagnet, reduces to that in Eq. (62) in an isotropic ferromagnet; the shape of surface of $(\mathbf{M}_1)_z$ does not change with velocity V_2, and the surface is symmetrical with respect to the center, as illustrated by Fig. 3.

(3) The depth and width of the surface of $(\mathbf{M}_1)_z$ are not constants but vary periodically with time. When $\eta\to 0$, the depth and width of the surface of $(\mathbf{M}_1)_z$, expressed by Eq. (62) in an isotropic ferromagnet, does not change periodically with time; the surface is also symmetrical with respect to the center, as illustrated by Fig. 3.

In terms of soliton solutions (55) and (56) in an anisotropic ferromagnet, we can find that the z component of the total magnetic moment

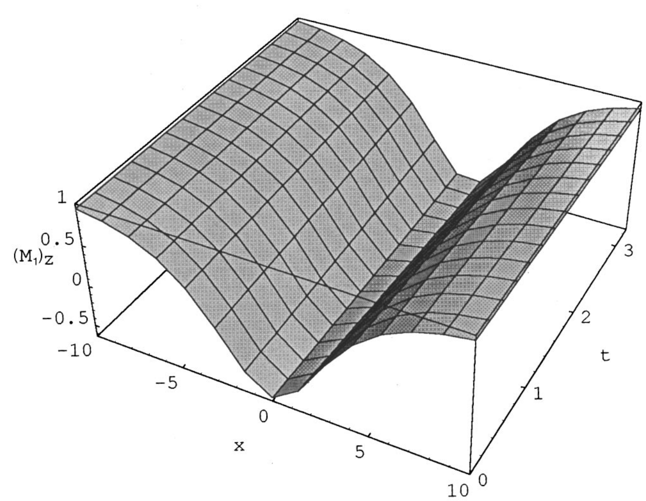

FIG. 2. Some graphical illustrations of a soliton solution $(\mathbf{M}_1)_z$ expressed by Eq. (54) in an anisotropic ferromagnet, where $\eta=0.33$, $\xi_1'=0.1$, $\xi_1''=0.2$, $x_{10}=0$, $x_{20}=0$, and $\pi/(4V_1)$ as units of time.

$$P_z=M_0\int dx(1-\cos\theta) \tag{65}$$

is not a constant, and it is dependent periodically on time, where P_z has the sense of the mean number of spin deviated from the ground state in a localized magnetic excitation. The total momentum of the magnetization field,

$$\mathbf{P}=-M_0\int dx(1-\cos\theta)\nabla\varphi, \tag{66}$$

is also not constant. Only in the case of an isotropic ferromagnet $\eta=0$ are the operators P_z and $\mathbf{P}$ constants of motion. Tjio and Wright [21] took advantage of this in solving the equation of motion. These properties are important for magnetization in an anisotropic ferromagnet with an external field, but they have never been obtained by other methods.

IV. ASYMPTOTIC BEHAVIOR OF MULTISOLITON SOLUTIONS

In this section we will construct a direct procedure for studying the asymptotic behavior of multisoliton solutions in an anisotropic ferromagnet with a magnetic field. According to Eq. (10), we can define

$$\Psi_N(\xi)=J_N(\xi)\Psi_0(\xi), \tag{67}$$

where

$$J_N(\xi)=D_N(\xi)D_{N-1}(\xi)\ldots D_1(\xi), \tag{68}$$

where $J_N(\xi)$ has N pairs of poles ξ_n and $-\overline{\xi}_n$, $n=1,2,\ldots,N$. Similar to Eq. (11), we can obtain the Lax equations for $\Psi_N(\xi)$,

$$\partial_x\Psi_N(\xi)=L_N(\xi)\Psi_N(\xi),$$
$$\partial_t\Psi_N(\xi)=A_N(\xi)\Psi_N(\xi). \tag{69}$$

On the basis of Eq. (12), $J_N(\xi)$ can be written as

$$J_N(\xi)=K_NP_N(\xi), \tag{70}$$

where

$$K_N(\xi)=C_N(\xi)C_{N-1}(\xi)\ldots C_1(\xi) \tag{71}$$

and

$$P_N(\xi)=I-\sum_{n=1}^{N}\frac{1}{\xi_n-\xi}G_n+\sum_{n=1}^{N}\frac{1}{\overline{\xi}_n+\xi}\widetilde{G}_n, \tag{72}$$

where K_N is a 2×2 matrix independent of ξ, i.e.,

$$K_N(\xi)=\exp\left[\frac{i}{2}\Theta_N(\xi)\sigma_z\right], \tag{73}$$

where

$$\Theta_N(\xi)=\sum_{n=1}^{N}\theta_n. \tag{74}$$

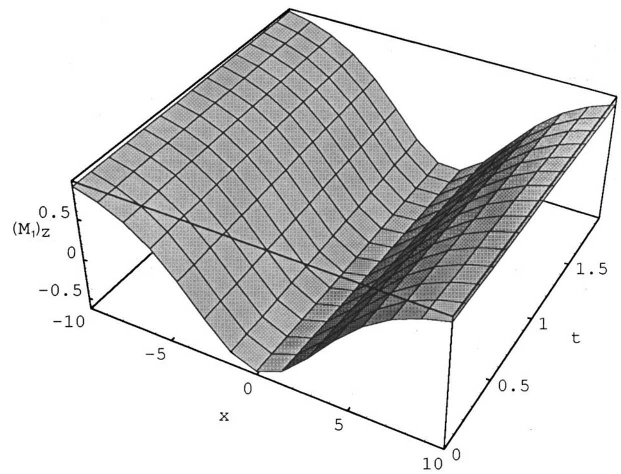

FIG. 3. Some graphical illustrations of a soliton solution $(\mathbf{M}_1)_z$ expressed by Eq. (62) in an isotropic ferromagnet, where $\sigma=0$, $\xi_1'=0.1$, $\xi_1''=0.2$, $x_{10}=0$, $x_{20}=0$, and $\pi/(4V_1)$ as units of time.

By means of Appendix A, we can obtain the relations

$$J_N(\xi)=\sigma_x\overline{J(-\overline{\xi})}\sigma_x, \tag{75}$$

$$J_N^\dagger(\overline{\xi})=J_N^{-1}\xi, \tag{76}$$

$$J_N(\xi)J_N^{-1}(\xi)=J_N^{-1}(\xi)J_N(\xi)=I, \tag{77}$$

$$P_N^\dagger(\overline{\xi})=I-\sum_{n=1}^{N}\frac{1}{\overline{\xi}_n-\xi}G_n^\dagger-\sum_{n=1}^{N}\frac{1}{\xi_n+\xi}\sigma_xG_n^T\sigma_x, \tag{78}$$

$$P_N^{-1}(\xi)=P_N^\dagger(\overline{\xi}), \tag{79}$$

and

$$\widetilde{G}_n=-\sigma_x\overline{G_n}\sigma_x. \tag{80}$$

Because $J_N(\xi)J_N^{-1}(\xi)=J_N^{-1}(\xi)J_N(\xi)=I$ in Eq. (77), its residue at $\xi=\xi_n$ should vanish, i.e., $G_mP_N^\dagger(\overline{\xi}_m)=0$, i.e.,

$$G_m\left(I-\sum_{n=1}^{N}\frac{1}{\overline{\xi}_n-\xi_m}G_n^\dagger-\sum_{n=1}^{N}\frac{1}{\xi_n+\xi_m}\sigma_xG_n^T\sigma_x\right)=0. \tag{81}$$

This result shows that G_m is degenerated; therefore G_n can be defined

$$G_n=(\alpha_n'\ \ \beta_n')^T(\gamma_n'\ \ \delta_n'). \tag{82}$$

In order to solve Eqs. (81) and (82), we will introduce a transformation

$$J_N'(\xi)=U^{-1}J_N(\xi)U \tag{83}$$

and

$$G_n'=U^{-1}G_nU,\ \ \widetilde{G}_n'=U^{-1}\widetilde{G}_nU=-\overline{G'}_n, \tag{84}$$

where $U^{-1}\sigma_x\overline{U}=i$.

Corresponding to Eq. (81), one can write

$$G'_m\left(I-\sum_{n=1}^{N}\frac{1}{\bar{\xi}_n-\xi_m}G_n'^{\dagger}-\sum_{n=1}^{N}\frac{1}{\xi_n+\xi_m}G_n'^{T}\right)=0. \quad (85)$$

Taking the limit $\xi\to\xi_n$ in Eq. (69), we can obtain

$$\partial_x(K_NG_n\Psi_0(\xi_n))=L_N(\xi_n)(K_NG_n\Psi_0(\xi_n)),$$
$$\partial_t(K_NG_n\Psi_0(\xi_n))=A_N(\xi_n)(K_NG_n\Psi_0(\xi_n)). \quad (86)$$

Since G_n is degenerate, the factor

$$(\gamma'_n \;\; \delta'_n)\Psi_0(\xi_n) \quad (87)$$

must be independent of x and t. Therefore, we can simply obtain

$$(\gamma'_n \;\; \delta'_n)=(b_n \;\; 1)\Psi_0^{-1}(\xi_n), \quad (88)$$

where b_n is a constant which has been shown in Eq. (36), while α'_n, β'_n, γ'_n, and δ'_n are different from α_n, β_n, γ_n, and δ_n except for $\gamma'_1=\gamma_1$ and $\delta'_1=\delta_1$.

Similar to Eq. (82), G'_n can be written as

$$G'_n=(\rho_n \;\; \nu_n)^T(f_n \;\; f_n^{-1}). \quad (89)$$

where

$$f_n=b_1^{1/2}\exp\left\{-i(\xi-\eta^2\xi^{-1})\left[x-2\frac{(\xi^2+\eta^2)^2}{\xi(\xi^2-\eta^2)}t\right]\sigma_z\right\}. \quad (90)$$

Substituting Eq. (89) into Eq. (85), we can obtain

$$f_m=\sum_{n=1}^{N}\frac{1}{\bar{\xi}_n-\xi_m}(f_m\bar{f}_n+f_m^{-1}\bar{f}_n^{-1})\bar{\rho}_n+\sum_{n=1}^{N}\frac{1}{\xi_n+\xi_m}(f_mf_n+f_m^{-1}f_n^{-1})\rho_n \quad (91)$$

and

$$f_m^{-1}=\sum_{n=1}^{N}\frac{1}{\bar{\xi}_n-\xi_m}(f_m\bar{f}_n+f_m^{-1}\bar{f}_n^{-1})\bar{\nu}_n+\sum_{n=1}^{N}\frac{1}{\xi_n+\xi_m}(f_mf_n+f_m^{-1}f_n^{-1})\nu_n. \quad (92)$$

By means of Eqs. (91) and (92), one can find ρ_n, ν_n, and $P'_N(\xi)$, e.g.,

$$P'_N(1)_{11}=1-\sum_{n=1}^{N}\frac{1}{\bar{\xi}_n+1}\overline{\rho_nf_n}-\sum_{n=1}^{N}\frac{1}{\xi_n-1}\rho_nf_n. \quad (93)$$

According to Eqs. (91) and (92), we can also obtain

$$1=\sum_{n=1}^{N}\frac{1}{\bar{\xi}_n-\xi_m}(1+f_m^{-2}\bar{f}_n^{-2})\overline{\rho_nf_n}+\sum_{n=1}^{N}\frac{1}{\xi_n+\xi_m}(1+f_m^{-2}f_n^{-2})\rho_nf_n \quad (94)$$

and

$$1=\sum_{n=1}^{N}\frac{1}{\bar{\xi}_n+\bar{\xi}_m}(1+\bar{f}_m^{-2}\bar{f}_n^{-2})\overline{\rho_nf_n}+\sum_{n=1}^{N}\frac{1}{\xi_n-\bar{\xi}_m}(1+\bar{f}_m^{-2}f_n^{-2})\rho_nf_n. \quad (95)$$

By means of Eqs. (94) and (95), ρ_n, $\bar{\rho}_n$, $P'_N(\xi)_{11}$, and $P'_N(\xi)_{12}$ can be easily determined. However, ρ_n and $\bar{\rho}_n$ will appear in every equation of Eqs. (94) and (95), and it is difficult to obtain explicit expressions of them by the well-known Binet-Cauchy formula. The asymptotic behaviors of the multisoliton solutions can be derived from them.

Introducing

$$\Delta_l=\begin{cases}\rho_nf_n & \text{if } n=l, \quad l\in 1,2,\ldots,N\\ \overline{\rho_nf_n} & \text{if } n=l-N, \quad l\in N+1,N+2,\ldots,2N\end{cases} \quad (96)$$

and

$$E_n=1, \quad l\in 1,2,\ldots,2N, \quad (97)$$

where E is a row matrix, Eqs. (94) and (95) can be expressed by

$$E=\Delta Q, \quad (98)$$

where Q is a $2N\times 2N$ matrix,

$$Q_{n,m}=\frac{1}{\xi_n+\xi_m}(1+f_n^{-2}f_m^{-2}), \quad (99)$$

$$Q_{n,N+m}=\frac{1}{\xi_n-\bar{\xi}_m}(1+f_n^{-2}\bar{f}_m^{-2}), \quad (100)$$

$$Q_{N+n,m}=\frac{1}{\bar{\xi}_n-\xi_m}(1+\bar{f}_n^{-2}f_m^{-2}), \quad (101)$$

$$Q_{N+n,N+m}=\frac{1}{\bar{\xi}_n+\bar{\xi}_m}(1+\bar{f}_n^{-2}\bar{f}_m^{-2}). \quad (102)$$

By means of Eq. (97), one can find

$$\Delta=EQ^{-1}. \quad (103)$$

$P'_N(1)_{11}$ in Eq. (93) can be written as

$$P'_N(1)_{11}=1+\sum_{l=1}^{2N}\Delta_lR_l=1+\Delta R^T, \quad (104)$$

where

$$R_l=\begin{cases}\dfrac{-1}{\xi_n-1} & \text{if } n=l, \quad l\in 1,2,\ldots,N\\[2ex] \dfrac{-1}{\bar{\xi}_n+1} & \text{if } n=l-N, \quad l\in N+1,N+2,\ldots,2N.\end{cases} \quad (105)$$

According to Eq. (97), $P'_N(1)_{11}$ in Eq. (104) can be expressed as

$$P_N'(1)_{11}=1+EQ^{-1}R^T=\frac{\det(Q+R^TE)}{\det Q}. \qquad (106)$$

When $N=1,\xi=\xi_j$, $\det Q$ is written as

$$\det Q=\det\begin{pmatrix} \frac{1}{2\xi_j}(1+f_j^{-4}) & \frac{1}{\xi_j-\bar{\xi}_j}(1+|f_j|^{-4}) \\ \frac{1}{\bar{\xi}_j-\xi_j}(1+|f_j|^{-4}) & \frac{1}{2\bar{\xi}_j}(1+\bar{f}_j^{-4}) \end{pmatrix}. \qquad (107)$$

By means of Eq. (90), f_n can be written as

$$f_n=\exp(-\phi_{1n}+i\phi_{2n}), \qquad (108)$$

where

$$\phi_{1n}=\frac{\xi_n''(|\xi_n|^2+\eta^2)}{|\xi_n|^2}(x-V_{1n}t-x_{1n0}) \qquad (109)$$

and

$$\phi_{2n}=\frac{\xi_n'(|\xi_n|^2-\eta^2)}{|\xi_n|^2}(x-V_{2n}t-x_{2n0}), \qquad (110)$$

while

$$V_{1n}=\frac{4\xi_n'(|\xi_n|^2-\eta^2)^2}{|\xi_n|^2}, \qquad (111)$$

$$V_{2n}=\frac{2(\xi_n'^2-\xi_n''^2)(|\xi_n|^4+\eta^2)^4+4\eta^4|\xi_n|^4}{\xi_n'|\xi_n|^2(|\xi_n|^2-\eta^2)}. \qquad (112)$$

Supposing all $\xi_n''>0$ and $V_{1N}>V_{1(N-1)}>\ldots>V_{11}$, the vicinity of $(V_{in}t-x_{1n0})$ $(i=1$ and 2) is denoted by Θ_n. In the extreme large t, these vicinities are separated from left to right as $\Theta_N,\Theta_{N-1},\ldots,\Theta_1$. In the vicinity Θ_j, there are limits,

$$(x-V_{1n}t-x_{1n0})\to-\infty, \quad |f_n|^{-1}\to 0 \quad \text{if } n<j \qquad (113)$$

and

$$(x-V_{1m}t-x_{1n0})\to\infty, \quad |f_m|^{-1}\to\infty \quad \text{if } m>j, \qquad (114)$$

while $\det Q$ tends to

$$\begin{vmatrix} \frac{1}{\xi_n+\xi_{n'}} & \frac{1}{\xi_n+\xi_j} & 0 & \frac{1}{\xi_n-\bar{\xi}_{n'}} & \frac{1}{\xi_n-\bar{\xi}_j} & 0 \\ \frac{1}{\xi_j+\xi_{n'}} & \frac{1+f_j^{-4}}{2\xi_j} & \frac{f_j^{-2}f_{m'}^{-2}}{\xi_j+\xi_{m'}} & \frac{1}{\xi_j-\bar{\xi}_{n'}} & \frac{1+|f_j|^{-4}}{\xi_j-\bar{\xi}_j} & \frac{f_j^{-2}\bar{f}_{m'}^{-2}}{\xi_j-\bar{\xi}_{m'}} \\ 0 & \frac{f_m^{-2}f_j^{-2}}{\xi_m+\xi_j} & \frac{f_m^{-2}f_{m'}^{-2}}{\xi_m+\xi_{m'}} & 0 & \frac{f_m^{-2}\bar{f}_j^{-2}}{\xi_m-\bar{\xi}_j} & \frac{f_m^{-2}\bar{f}_{m'}^{-2}}{\xi_m-\bar{\xi}_{m'}} \\ \frac{1}{\bar{\xi}_n-\xi_{n'}} & \frac{1}{\bar{\xi}_n-\xi_j} & 0 & \frac{1}{\bar{\xi}_n+\bar{\xi}_{n'}} & \frac{1}{\bar{\xi}_n+\bar{\xi}_j} & 0 \\ \frac{1}{\bar{\xi}_j-\xi_{n'}} & \frac{1+|f_j|^{-4}}{\bar{\xi}_j-\xi_j} & \frac{\bar{f}_j^{-2}f_{m'}^{-2}}{\bar{\xi}_j-\xi_{m'}} & \frac{1}{\bar{\xi}_j+\bar{\xi}_{n'}} & \frac{1+\bar{f}_j^{-4}}{2\bar{\xi}_j} & \frac{\bar{f}_j^{-2}\bar{f}_{m'}^{-2}}{\bar{\xi}_j+\bar{\xi}_{m'}} \\ 0 & \frac{\bar{f}_m^{-2}f_j^{-2}}{\bar{\xi}_m-\xi_j} & \frac{\bar{f}_m^{-2}f_{m'}^{-2}}{\bar{\xi}_m-\xi_{m'}} & 0 & \frac{\bar{f}_m^{-2}\bar{f}_j^{-2}}{\bar{\xi}_m+\bar{\xi}_j} & \frac{\bar{f}_m^{-2}\bar{f}_{m'}^{-2}}{\bar{\xi}_m+\bar{\xi}_{m'}} \end{vmatrix}, \qquad (115)$$

where $n,n'<j<m,m'$.

In Appendix C, we find that the asymptotic behavior of the multisoliton solutions in limits (113) is similar to the single soliton solution, but f_j is replaced by $f_j^{(+)}$

$$f_j^{(+)}=\left(\frac{\tau_j}{\chi_j}\right)^{1/2}f_j, \qquad (116)$$

$$\tau_j=\prod_{n=1}^{j-1}\frac{(\xi_j-\xi_n)(\xi_j+\bar{\xi}_n)}{(\xi_j+\xi_n)(\xi_j-\bar{\xi}_n)}, \qquad (117)$$

$$\chi_j=\prod_{m=j+1}^{N}\frac{(\xi_j-\xi_m)(\xi_j+\bar{\xi}_m)}{(\xi_j+\xi_m)(\xi_j-\bar{\xi}_m)}, \qquad (118)$$

while $\det Q\to\det Q_j^{(+)}$

$$\det Q_j^{(+)}=-\frac{(\xi_j-\bar{\xi}_j)^2}{4|\xi_j|^2|\xi_j-\bar{\xi}_j|^2}(1+|f_j^{(+)}|^{-8}) + \frac{1}{4|\xi_j|^2}[(f_j^{(+)})^{-4}+(\overline{f_j^{(+)}})^{-4}], \qquad (119)$$

the asymptotic expression of detQ' should be obtained. Meanwhile, $\Phi_{1j}^{(+)}$ and $\Phi_{2j}^{(+)}$, corresponding to those in Eqs. (109) and (110), can be written as

$$\Phi_{1j}^{(+)}=\frac{\xi_j'(|\xi_j|^2-\eta^2)}{|\xi_j|^2}(x-V_{1j}t-x_{1j0}-\Gamma_{1j}^{(+)}), \quad (120)$$

$$\Phi_{2j}^{(+)}=\frac{\xi_j\prime(|\xi_j|^2-\eta^2)}{|\xi_j|^2}(x-V_{2j}t-x_{2j0}-\Gamma_{2j}^{(+)}), \quad (121)$$

where

$$\Gamma_{1j}^{(+)}=\frac{1}{2\lambda_j'}(\ln|\tau_j|-\ln|\chi_j|), \quad (122)$$

$$\Gamma_{2j}^{(+)}=\arg\tau_j-\arg\chi_j. \quad (123)$$

Similarly, when $t\to-\infty$, the asymptotic behavior of multisoliton solutions in the vicinity of Θ_j can be written, e.g., analogous to Eqs. (120) and (121),

$$\Gamma_{1j}^{(-)}=-\Gamma_{1j}^{(+)}, \quad (124)$$

$$\Gamma_{2j}^{(-)}=-\Gamma_{2j}^{(+)}; \quad (125)$$

therefore, the total additional displacement of Γ_{1j} and the total phase shift Γ_{2j} are

$$\Gamma_{1j}=2\Gamma_{1j}^{(+)}, \quad (126)$$

$$\Gamma_{2j}=2\Gamma_{2j}^{(+)}. \quad (127)$$

V. CONCLUSION

In the present paper we introduced a particular parameter ξ for equations of motion of the magnetization in an anisotropic ferromagnet with a magnetic field; the Lax equations for the Darboux matrices are generated recursively. By choosing the constants involved in the Darboux matrices, the Jost solutions satisfy the corresponding Lax equations, the exact soliton solutions describing nonlinear dynamics of the magnetization are investigated, and the asymptotic behavior of multisoliton solutions are also analyzed. These results have never been found by any other methods. They may be useful for further theoretical research and practical application.

Equations (52)–(54) show that the soliton solutions in an anisotropic ferromagnet depend essentially on two velocities, V_1 in Eq. (50) and V_2 in Eq. (51). The center of an inhomogeneity moves with a constant velocity V_1, while the shape of the solitary waves also changes with another velocity V_2. Therefore, the depths and widths of the surface of the solitary waves are not constants but vary periodically with time, and the shape of the solitary waves is not symmetrical with respect to the center. By means of these features, we find that soliton solutions in an anisotropic ferromagnet cannot be expressed in the form of product of separated variables in the moving coordinates [21,22]. Only when $\eta\to 0$ can these soliton solutions in an anisotropic ferromagnet reduce to those in an isotropic ferromagnet; for example, the single soliton solutions (63) and (64) in the polar coordinates are equivalent to Eq. (22) obtained by means of the method of separating variables in the moving coordinates in Ref. [21]. Therefore, it is impossible to investigate the exact soliton solutions in an anisotropic ferromagnet by means of the method of separating variables.

Reducing the equations of motion to an appropriate form, Kosevich, Ivanov, and Kovalev [29] found a solution. In terms of Eq. (55) in the polar coordinates, there exists

$$\tan^2\left(\frac{\theta}{2}\right)=\frac{\xi_1''^2(|\xi_1|^2-\eta^2)^2+4\eta^2\xi_1''^2|\xi_1|^2\sin^2\phi_2}{|\xi_1|^2(|\xi_1|^2-\eta^2)^2\cosh^2\phi_1-\xi_1''^2(|\xi_1|^2-\eta^2)^2}. \quad (128)$$

If we compared Eq. (128) with an approximate solution given in Ref. [29], we can find that previous properties of the soliton solutions remain even in an approximation on the order of η^2. The solutions of Ref. [29] do not satisfy the Landau-Lifschitz equation for an anisotropic ferromagnet even in the first order of anisotropy, and there is no reason to consider it as an approximate solution; all attempts tried in this approximation were not successful.

Introducing a particular parameter ξ in Eq. (6), while $\xi=\pm\eta$ corresponds to zero λ and to $\mu=\pm 2\eta$. In the complex μ plane, these two points are the edges of cuts. ξ contributes to the determination factor C_n in Eq. (12). C_n is important to ensure that the Jost solution generated satisfies the corresponding Lax equations. This indicates that in the inverse scattering transformation the edges of cuts in the complex plane must make a contribution even in the case of nonreflection. Unfortunately, Borovik and Kulinich [33,34] apparently did not consider these effects. Evidently, they did not obtain any expression of the solution.

Using the Hirota method, Bogdan and Kovalev [35] sought soliton solutions of the Landau-Lifshitz equation in an anisotropic ferromagnet in the form

$$\mathbf{M}_x+i\mathbf{M}_y=\frac{2fig}{|f|^2+|g|^2}, \quad \mathbf{M}_z=\frac{|f|^2-|g|^2}{|f|^2+|g|^2}, \quad (129)$$

while

$$f=\sum_{n=0}^{[N/2]}\sum_{C_2n} a(i_1,\ldots,i_{2n})\exp(\rho_{i_1}+\cdots+\rho_{i_{2n}}), \tag{130}$$

$$g^*=\sum_{m=0}^{[(N-1)/2]}\sum_{C_{2m+1}} a(j_1,\ldots,j_{2m+1})\times\exp(\rho_{j_1}+\cdots+\rho_{j_{2m+1}}), \tag{131}$$

where $[N/2]$ is the maximum integer in addition to $N/2$, C_n represents a summation over all combinations of N elements in n, and $\rho_i=(k_i+\omega_i t+\rho_i^0)$, while

$$a(i_1,\ldots,i_n)=\begin{cases}\sum_{k<l}^{(n)} a(i_k,i_l) & \text{for } n\geqslant 2\\ 1 & \text{for } n=0,1.\end{cases} \tag{132}$$

According to the expression of the single soliton solutions (52)–(54) in this paper, one can see that they are difficult to be expressed in the form of Hirota factorization. Obviously, Bogdan and Kovalev [35] did not obtain the desired results.

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APPENDIX A: SOME RELATIONS

According to Eqs. (4), (5), and (6), we can obtain the relations

$$\lambda(-\overline{\xi})=-\overline{\lambda(\xi)}, \qquad \mu(-\overline{\xi})=-\overline{\mu(\xi)} \tag{A1}$$

and

$$L(-\overline{\xi})=\sigma_x\overline{L(\xi)}\sigma_x, \qquad L^\dagger(\overline{\xi})=-L(\xi) \tag{A2}$$

and

$$A(-\overline{\xi})=\sigma_x\overline{A(\xi)}\sigma_x, \qquad A^\dagger(\overline{\xi})=-A(\xi). \tag{A3}$$

In terms of Eq. (9), there exist

$$\Psi_0(-\overline{\xi})=-i\sigma_x\overline{\Psi_0(\xi)}, \qquad \Psi_0^\dagger(\overline{\xi})=\Psi_0^{-1}(\xi); \tag{A4}$$

then, using these relations, one can find

$$\Psi_n(-\overline{\xi})=-i\sigma_x\overline{\Psi_n(\xi)}, \qquad \Psi_n^\dagger(\overline{\xi})=\Psi_n^{-1}(\xi) \tag{A5}$$

and

$$D_n(-\overline{\xi})=\sigma_x\overline{D_n(\xi)}\sigma_x, \qquad D_n^\dagger(\overline{\xi})=D_n^{-1}(\xi). \tag{A6}$$

APPENDIX B: DETERMINATION OF F_n AND $\widetilde{F}_N$

Putting

$$F_n=(\alpha_n\beta_n)^T(\gamma_n\delta_n), \tag{B1}$$

then substituting it into Eq. (21), we can obtain the following linear equations:

$$\gamma_n-(|\gamma_n|^2+|\delta_n|^2)\overline{\alpha_n}-\frac{\xi_n-\overline{\xi_n}}{\xi_n}\gamma_n\delta_n\beta_n=0, \qquad \delta_n-(|\gamma_n|^2+|\delta_n|^2)\overline{\beta_n}-\frac{\xi_n-\overline{\xi_n}}{\xi_n}\gamma_n\delta_n\alpha_n=0. \tag{B2}$$

Using γ_n and δ_n to express α_n, β_n, F_n, and $\widetilde{F}_n$ can be written as

$$F_n=\xi[(\overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2)(\overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2)]^{-1}\begin{pmatrix}\overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2 & 0\\ 0 & \overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2\end{pmatrix}\times\begin{pmatrix}\overline{\gamma_n}\\ \overline{\delta_n}\end{pmatrix}(\gamma_n \quad \delta_n) \tag{B3}$$

and

$$\widetilde{F}_n=\overline{\xi}[(\overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2)(\overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2)]^{-1}\begin{pmatrix}\overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2 & 0\\ 0 & \overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2\end{pmatrix}\times\begin{pmatrix}\gamma_n\\ \delta_n\end{pmatrix}(\overline{\gamma_n} \quad \overline{\delta_n}). \tag{B4}$$

APPENDIX C: ANALYSIS OF ASYMPTOTIC BEHAVIOR OF THE MULTISOLITON SOLUTIONS

In Eq. (115), only those terms leading to $|f_{j+1}|^{-8}\cdots|f_N|^{-8}$ remain, and it is difficult to calculate this determinant. Similar to the procedure in Ref. [34], we will only consider the term without f_j,

$$\left|\begin{matrix} \frac{1}{\xi_n+\xi_{n'}} & \frac{1}{\xi_n+\xi_j} & 0 & \frac{1}{\xi_n-\overline{\xi_{n'}}} & \frac{1}{\xi_n-\overline{\xi_j}} & 0 \\ \frac{1}{\xi_j+\xi_{n'}} & \frac{1}{2\xi_j} & 0 & \frac{1}{\xi_j-\overline{\xi_{n'}}} & \frac{1}{\xi_j-\overline{\xi_j}} & 0 \\ 0 & 0 & \frac{f_m^{-2}f_{m'}^{-2}}{\xi_m+\xi_{m'}} & 0 & 0 & \frac{f_m^{-2}\overline{f_{m'}^{-2}}}{\xi_m-\overline{\xi_{m'}}} \\ \frac{1}{\overline{\xi_n}-\xi_{n'}} & \frac{1}{\overline{\xi_n}-\xi_j} & 0 & \frac{1}{\overline{\xi_n}+\overline{\xi_{n'}}} & \frac{1}{\overline{\xi_n}+\overline{\xi_j}} & 0 \\ \frac{1}{\overline{\xi_j}-\xi_{n'}} & \frac{1}{\overline{\xi_j}-\xi_j} & 0 & \frac{1}{\overline{\xi_j}+\overline{\xi_{n'}}} & \frac{1}{2\overline{\xi_j}} & 0 \\ 0 & 0 & \frac{\overline{f_m^{-2}}f_{m'}^{-2}}{\overline{\xi_m}-\xi_{m'}} & 0 & 0 & \frac{\overline{f_m^{-2}}\,\overline{f_{m'}^{-2}}}{\overline{\xi_m}+\overline{\xi_{m'}}} \end{matrix}\right|. \tag{C1}$$

The term involving f_j^{-4} is the following determinant:

$$\left|\begin{matrix} \frac{1}{\xi_n+\xi_{n'}} & 0 & 0 & \frac{1}{\xi_n-\xi_{n'}} & \frac{1}{\xi_n-\overline{\xi_j}} & 0 \\ 0 & \frac{f_j^{-4}}{2\xi_j} & \frac{f_j^{-2}f_{m'}^{-2}}{\xi_j+\xi_{m'}} & 0 & 0 & \frac{f_j^{-2}\overline{f_{m'}^{-2}}}{\xi_j-\overline{\xi_{m'}}} \\ 0 & \frac{f_m^{-2}f_j^{-2}}{\xi_m+\xi_j} & \frac{f_m^{-2}f_{m'}^{-2}}{\xi_m+\xi_{m'}} & 0 & 0 & \frac{f_m^{-2}\overline{f_{m'}^{-2}}}{\xi_m-\overline{\xi_{m'}}} \\ \frac{1}{\overline{\xi_n}-\xi_{n'}} & 0 & 0 & \frac{1}{\overline{\xi_n}+\overline{\xi_{n'}}} & \frac{1}{\overline{\xi_n}+\overline{\xi_j}} & 0 \\ \frac{1}{\overline{\xi_j}-\xi_{n'}} & 0 & 0 & \frac{1}{\overline{\xi_j}+\overline{\xi_{n'}}} & \frac{1}{2\overline{\xi_j}} & 0 \\ 0 & \frac{\overline{f_m^{-2}}f_j^{-2}}{\overline{\xi_m}-\xi_j} & \frac{\overline{f_m^{-2}}f_{m'}^{-2}}{\overline{\xi_m}-\xi_{m'}} & 0 & 0 & \frac{\overline{f_m^{-2}}\,\overline{f_{m'}^{-2}}}{\overline{\xi_m}+\overline{\xi_{m'}}} \end{matrix}\right|. \tag{C2}$$

In addition to the common factor $|f_{j+1}|^{-8}\dots|f_N|^{-8}$, these two determinants are clearly proportional to

$$\left|\begin{matrix} \frac{1}{\xi_n+\xi_{n'}} & \frac{1}{\xi_n-\overline{\xi_{n'}}} & \frac{1}{\xi_n-\overline{\xi_j}} \\ \frac{1}{\overline{\xi_n}-\xi_{n'}} & \frac{1}{\overline{\xi_n}+\overline{\xi_{n'}}} & \frac{1}{\overline{\xi_n}+\overline{\xi_j}} \\ \frac{1}{\overline{\xi_n}-\xi_{n'}} & \frac{1}{\xi_n+\overline{\xi_{n'}}} & \frac{1}{2\overline{\xi_j}} \end{matrix}\right| \left|\begin{matrix} \frac{1}{\xi_m+\xi_{m'}} & \frac{1}{\xi_m-\overline{\xi_{m'}}} \\ \frac{1}{\overline{\xi_m}-\xi_{m'}} & \frac{1}{\overline{\xi_m}+\overline{\xi_{m'}}} \end{matrix}\right|; \tag{C3}$$

the proportional coefficients are

$$\frac{(\xi_j+\overline{\xi_j})^2}{2\xi_j|\xi_j-\overline{\xi_j}|^2}\prod_{n=1}^{j-1}\frac{(\xi_j-\xi_n)^2(\xi_j+\overline{\xi_n})^2}{(\xi_j+\xi_n)^2(\xi_j-\overline{\xi_n})^2} \tag{C4}$$

and

$$-\frac{1}{2\xi_j}\prod_{m=j+1}^{N}\frac{(\xi_j-\xi_m)^2(\xi_j+\overline{\xi_m})^2}{(\xi_j+\xi_m)^2(\xi_j-\overline{\xi_m})^2}. \tag{C5}$$

Therefore, the asymptotic behavior of the multisoliton solutions in the limits (113) and (114) is similar to the single soliton solution.

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